Indian Statistical Institute, Bangalore Centre

Algebraic Topology (B. Math (Hons.) III year)

Final Exam (50 MARKS)

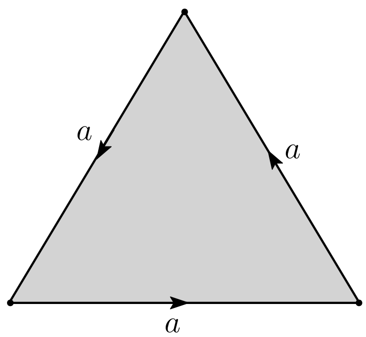
Date: 19/11/2024 TIME ALLOWED: 3 HOURS

Instructions

- **Answer all the questions from Part–A (Questions 1 to 6).**
- **Answer ONLY TWO questions from Part–B (Questions 7, 8, and 9) each carries 10 MARKS.**
- $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.
- S^n denotes the n-dimensional sphere.
- $\mathbb{R}P^2$ denotes the real projective plane.
- T^2 denotes the $2-$torus $S^1 \times S^1$.

Part–A

Question 1. The $m-$fold dunce cap Γ_m is the quotient of the closed unit disk obtained by identifying each point on the boundary circle with its translates under rotation by $\frac{2\pi}{m}$; specifically, for each θ, the points $e^{i(\theta+\frac{2\pi r}{m})}$, for $r = 0, 1, ..., m-1$ are identified (but keep the interior of the disk unchanged). Refer Figure for $3-$fold dunce cap Γ_3.



3−fold Dunce Cap

Find homology groups of the $3-$fold dunce cap Γ_3. (6 Marks)

Question 2. Let $f : S^n \to S^n$ be a continuous map. Suppose that f is not onto, then show that $deg(f) = 0$. (3 Marks)

Question 3. Let $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ be a continuous map such that the induced homomorphism $f_* : H_1(\mathbb{R}P^2) \to H_1(\mathbb{R}P^2)$ is an isomorphism. Prove that f is surjective. (4 Marks)

Question 4. Prove that $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^n$, for $n \in \mathbb{N}$ if and only if $n = 2$. (4 Marks)

Question 5. Let X be a space consisting of two spheres S^n and S^m that are tangent at a point, where $m, n \geq 2$, i.e., $X = S^n \vee S^m$, the wedge sum of S^n and S^m. Find the fundamental group $\pi_1(X)$ of X. (3 Marks)

Question 6. List all the covering spaces (E, p) of the Möbius band M up to covering space isomorphism. Justify your answer. (10 Marks)

Part–B

Question 7. Let X be a topological space and B a subspace of X. If there is a continuous map $f : X \to B$ which leaves each point of B fixed, then B is called a retract of X. The function f is called a retraction of X onto B.

(i) Let A and K be complexes for which A is a subset of K and $|A|$ is a retract of $|K|$. Prove that $H_p(K)$ has a subgroup isomorphic to $H_p(A)$ in each dimension p.

(4 Marks)

(ii) Prove that there is no retraction from the Möbius band to its boundary circle.

(4 Marks)

(iii) If K is a $2-$pseudo-manifold then prove that the Euler characteristic $\chi(K) \leq 2$.

(2 Marks)

Question 8. Let X and Y be topological spaces. Let $f : X \to Y$ be a continuous map. Define the function $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(y_0))$ by $f_*([\alpha]) = [f \circ \alpha]$, for $[\alpha] \in \pi_1(X, x_0)$.

(i) Prove that f_* is a well-defined group homomorphism. Moreover, if f is a homeomorphism then the map f_* is an isomorphism. (6 Marks)

(ii) Determine the number of path-connected coverings of the space $\mathbb{R}P^2 \times \mathbb{R}P^2$ (up to equivalence of coverings). Justify your answer. (4 Marks)

Question 9. Let S^2 be a 2−sphere and let T^2 be a 2−torus $S^1 \times S^1$.

(i) Prove that any continuous map from a $2-$Sphere S^2 to a $2-$Torus T^2 is homotopic to a constant map. (6 Marks)

(ii) Prove or disprove: There are exactly 4 non-isomorphic (as a covering space) $2-$sheeted covering spaces of the torus T^2. (4 Marks)

END OF PAPER